Decomposing portfolio risk using Monte Carlo estimators

D. Mossessian, *dmossessian@factset.com*
V. Vieli, *vvieli@factset.com*

Decomposing portfolio risk using Monte Carlo estimators

Contents

Introduction . . . 2
Risk measures, marginal risk contributions and risk measure components . . . 2
Monte Carlo estimates of VaR and marginal contributions to VaR . . . 3
Monte Carlo estimators of component risk in MAC model . . . 5
  Portfolio VaR estimator . . . 6
  Security contribution to VaR . . . 6
  Factor Contribution to VaR . . . 7
  Security and Factor contribution . . . 7
Choice of ε and numerical examples . . . 8
Conclusion . . . 9
Bibliography . . . 9
Appendix . . . 11
Marginal contribution to VaR as conditional expectation . . . 11
Security and factor correction parameters . . . 11

Introduction

Investment practitioners rely on various risk measures - Tracking Error Volatility (TEV), Value-at-Risk (VaR), Expected Shortfall (ES) - and their decomposition into contributions from securities and risk factors to guide investment decisions. In this note we analyze decomposition of portfolio risk into marginal contributions (measuring individual component impact on total risk), or additive components (the components that will sum to the total portfolio risk measure), each component being associated with a single security, an investment sector, an asset class, or a risk factor. The key property underlying this decomposition is positive homogeneity of the risk measure, a property that states that the risk of a portfolio scales in proportion to the size of the portfolio. Taking this property into account, these risk measure decompositions are easily computed under the assumption of joint normality of the component returns. In most of actual cases, however, the assumption of normally distributed returns does not apply. Fat tailed distributions are rule rather than exception for financial market factors and the inclusion of non-linear derivative instruments in the portfolio gives rise to distributional asymmetries. Whenever these deviations from normality are expected to cause serious biases in VaR calculations, one has to resort to either alternative distribution specifications or simulation methods. In this note we present the methodology for estimating these metrics in the context of Monte Carlo simulations.

The marginal risk contributions and components associated with both Value at Risk and Expected Shortfall can be represented as conditional expectations of component returns, conditioned on events in the tail of the loss distribution for the full portfolio. The rarity of these tail events presents an obstacle to practical calculation of these conditional expectations. Each contribution depends on the probability of a rare event (a large loss of a particular component) conditional on an even rarer event (an extreme loss for the portfolio as a whole). This note describes methods and techniques we use to address the practical difficulties of calculating these expectations.

Risk measures, marginal risk contributions and risk measure components

The distribution of returns in a portfolio is typically summarized through a scalar measure of risk. Two of the most commonly used risk measures are Value-at-Risk (VaR) and Expected Shortfall (ES). The VAR associated with probability $1-\alpha$ (eg, $\alpha = 1\%$) is the lower bound for the loss incurred by a portfolio with probability α:

$$VaR_\alpha : P(r_p \leq -VaR_\alpha) = \alpha \tag{1}$$

where r_p is the portfolio return. The corresponding expected shortfall is the conditional expectation

$$ES_\alpha = \mathbb{E}[r_p | r_p \leq -VAR_\alpha] \tag{2}$$

and can be intuitively interpreted as the average of all losses above a given quantile of the loss distribution.

Value-at-Risk is in more widespread use, but expected shortfall is coherent (in the sense of [1]) while VaR is not. VaR is not in general subadditive, which means that the sum of the VaRs for two portfolios is not equal to the VaR for the combined portfolio. In particular this means that the VaR of a portfolio cannot be decomposed as a sum of the standalone VaRs of its components.

For the purpose of risk management it is not sufficient to just estimate a single measure of the portfolio risk as a whole. For capital allocation, measurement of risk-adjusted performance, developing hedging strategies and in general understanding the impact of different risk factors and component on portfolio risk it is useful to allocate the risk to elements of the portfolio based on their marginal contribution to total risk. To see how the marginal contribution to VaR can be calculated let us consider the return r_p of a portfolio that consists of n securities, each having return r_i, with the weight of the i^{th} security in the portfolio denoted as w_i:

$$r_p = \sum_{i=1}^{n} w_i r_i \tag{3}$$

We should note that from the point of view of risk factor model, the return of the portfolio can be similarly represented as a weighted sum of the factor returns. Thus, equation (3) can be used to look at the portfolio risk decomposition by factor returns, as well as by security returns.

The marginal contribution to portfolio VaR from component i, $MVaR_{\alpha,i}$, is the change in portfolio VaR resulting from a marginal change in the i^{th}component position :

$$MVaR_{\alpha,i} = \frac{\partial VaR_\alpha}{\partial w_i} \tag{4}$$

This metric allows portfolio managers to find the components that can be used to significantly revise the overall risk of the portfolio with the minimal change to capital allocation.

We can us the marginal contributions to VaR to define the additive VaR components that will sum to the total portfolio risk as $VaR_\alpha = \sum_i CVaR_{\alpha,i}$. To do that we should note that VaR defines a quantile of the portfolio return distribution and, as portfolio return, is a homogeneous function of component weights (meaning that multiplying all weights by the same number leads to VaR scaling by that number). Thus, according to Euler homogeneous function theorem [2], VaR can be decomposed as

$$VaR_\alpha = \sum_{i}^{n} w_i \frac{\partial VaR_\alpha}{\partial w_i} \tag{5}$$

This means that we can define an additive component of VaR in terms of the marginal contribution to VaR as

$$CVaR_{\alpha,i} = w_i MVaR_{\alpha,i} \tag{6}$$

It can be shown (see Appendix) that the marginal contribution to portfolio VaR is the conditional expectation of the component return, conditioned on rare values of the portfolio return VaR_α:

$$MVaR_{\alpha,i} = \frac{\partial VaR_\alpha}{\partial w_i} = -\mathbb{E}[r_i | r_p = -VaR_\alpha] \tag{7}$$

When VaR is estimated using a linear normal model, calculating contribution to VaR is fast and easy - one just need to use equation (4) and differentiate the parametric expression for VaR. But in case of Monte Carlo simulation contribution to VaR has some severe problems - the sampling variability of the estimate is large and will not go down as we increase the number of samples for the simulation. The problem is that the contribution to VaR from a given component depends on the single return sample that happens to be the α^{th} return observation for the portfolio (the simulated VaR_α). The contribution to VaR depends on that single return observation in such a way that the sampling variability does not change with the number of trials in the simulation. Next section describes Monte Carlo techniques used to compute VaR, explains in more details the problem with estimating VaR components and outlines the methodology used to overcome the problem (for detailed theoretical treatment of the problem and analysis of solution methods see, for example, [3],[4], and [5]).

Monte Carlo estimates of VaR and marginal contributions to VaR

Estimation of the risk decomposition described by (4) and (6) by Monte Carlo is a two steps procedure. First, the Value-at-Risk (and Expected Shortfall) is estimated, and then the risk contributions are computed using the value of VaR from the first step in place of the true VaR in the conditional expectations (7).

To create Monte Carlo estimator for portfolio VaR we should first write the equation (11) for VaR through the confidence level α as

$$\alpha = P(x \le -VaR_\alpha) = \int_{-\infty}^{-VaR_\alpha} f_r(x)dx \tag{8}$$

and note that this equation can be rewritten as an expectation of the indicator function defined as

$$\mathcal{I}(x \le a) = \begin{cases} 1 & \text{if } x \le a \\ 0 & \text{if } x > a \end{cases} \tag{9}$$

as

$$\alpha = \int_{-\infty}^{\infty} \mathcal{I}(x \le -VaR_\alpha) f_r(x)dx = \mathbb{E}_r[\mathcal{I}(x \le -VaR_\alpha)] \tag{10}$$

The expectation representation can be used to compute VaR when we generate, using Monte Carlo method, a sample of independent and identically distributed portfolio returns $r_i,\ i = 1, ..N$. In this case the estimator for the expectation is:

$$\alpha = \frac{1}{N}\sum_{i=1}^{N} \mathcal{I}(r_i \le -VaR_\alpha) \tag{11}$$

and in this form it can be used to find VaR from a sorted list of sample returns given a value of α. Suppose we pick a value of VaR equal to the n^{th} return in the sample $VaR_\alpha = -r_n$. Then for every sample with $i \le n$ in the sorted list the indicator function is equal one, and for every sample with $i > n$ the indicator function is zero. Thus, the confidence level for that value of VaR is

$$\alpha_{VaR} = \frac{n}{N} \tag{12}$$

So we can just find the value n in the list such that $\frac{n}{N}$ is closest to the given value of α and use the sample $-r_n$ as an estimate of VaR.

When estimating VaR of a portfolio that consists of S securities we are generating K i.i.d. vectors of security returns $\mathbf{r}_i^T = (r_{1i}, r_{2i}, ..., r_{Si})$ where $\mathbf{r}_i$ is a single (i^{th}) sample of a joint distribution of individual security returns r_{ji}. From these vectors we compute K i.i.d. portfolio returns $r_{ip} = \mathbf{w}^T \mathbf{r}_i$, and use these samples to estimate portfolio VaR.

An estimator for a conditional expectation (7) of the k^{th} security return in this case will be

$$\mathbb{E}[r_k | r_p = -VaR_\alpha] = \frac{\sum_{i=1}^{K} r_{ki} \mathcal{I}(r_{ip} = -VaR_\alpha)}{\sum_{i=1}^{N} \mathcal{I}(r_{ip} = -VaR_\alpha)} \tag{13}$$

Unfortunately, if we just generate a single sample of K values of r_{ip} the sums in the estimator for conditional expectation will only have a single non-zero term. In other words, we will have a single MC sample in the region of interest.

We can remedy this situation by generating a number of samples of K portfolio returns, such that we will have multiple realizations of $r_{ip} = VaR_\alpha$. This, however, is extremely inefficient. Instead, we can relax the condition in the expectation (13) from

$$r_p = -VaR_\alpha \tag{14}$$

to

$$\left| r_p + VaR_\alpha \right| < \varepsilon \tag{15}$$

so that the estimator of the marginal contribution to VaR becomes:

$$\widehat{MVaR}_{\alpha,k} = -\mathbb{E}[r_k | \left| r_p + VaR_\alpha \right| < \varepsilon] = -\frac{\sum_{i=1}^{K} r_{ki} \mathcal{I}(\left| r_{ip} + VaR_\alpha \right| < \varepsilon)}{\sum_{i=1}^{N} \mathcal{I}(\left| r_{ip} + VaR_\alpha \right| < \varepsilon)} \tag{16}$$

(here the symbol $\widehat{}$ indicates that this is a biased estimator for marginal contribution).

The size of the neighborhood ε will determine the number of active points in the estimator (16). We need to have a reasonable number of points in the neighborhood of VaR to bring the variance of the estimator down, but at the same time we have to restrict the width of the region to limit the variability of the portfolio return within the neighborhood.

Because the averaging region of formula (16) $\left| r_p + VaR_\alpha \right| < \varepsilon$ is located in the tail of the portfolio returns distribution, the median of the samples in the region will be less negative than the mean, and the weighted average of the conditional mean returns will be less negative than the portfolio quantile return. In other words, the weighted sum of marginal contribution estimators is expected to be less than the estimated portfolio VaR:

$$\sum_{i}^{n} w_i \widehat{MVaR}_{\alpha,i} < VaR_\alpha \tag{17}$$

To correct that we introduce the normalization factor ω defined as

$$\omega = \frac{VaR_\alpha}{\sum_i^n w_i \widehat{MVaR}_{\alpha,i}} \tag{18}$$

and define the adjusted estimator for marginal contribution to VaR as

$$MVaR_{\alpha,i} = \omega \widehat{MVaR}_{\alpha,i} \tag{19}$$

The corresponding estimate for the CVaR follows from eq.(6). Due to the adjustment factor ω the sum of the CVaRs exactly equals the initially estimated overall portfolio VaR, as required by CVaR definition.

Introduction of the normalization factor ω is very similar in nature to the method of control variates in Monte Carlo estimations. The method relies on knowing the expectation of an auxiliary simulated random variable, called a control. The known expectation is compared with the estimated expectation obtained by simulation. The observed discrepancy between the two is then used to adjust estimates of other (unknown) quantities that are the primary focus of the simulation. In our case the portfolio VaR is used as a control variate for component VaR estimators. The more detailed analysis of the adjusted estimator and justification of the normalization procedure can be found in [4].

Monte Carlo estimators of component risk in MAC model

For a portfolio of S securities we generate K Monte Carlo samples (typically $K = 5000$) for each of the F factors. As a result we have a 3D matrix of returns for each security, each factor, each sample. We can visualize the matrix as having F vertical slices, each slice is a matrix of S rows and K columns. Each row of the slice matrix is the set of returns for one security/one factor, obtained on K Monte Carlo samples.

That is, each row of each slice of the matrix is a distribution of returns from one security/one factor. If we add all vertical slices together, we will obtain the $S \times K$ matrix, where each row will be a distribution of returns of individual security. If we slice the 3D matrix horizontally, instead of vertically, (imagine horizontal 2D matrices stacked on top of each other), multiply each horizontal slice by the appropriate security weight and add the slices together, we will have $F \times K$ matrix where each row is the distribution of returns from individual factor.

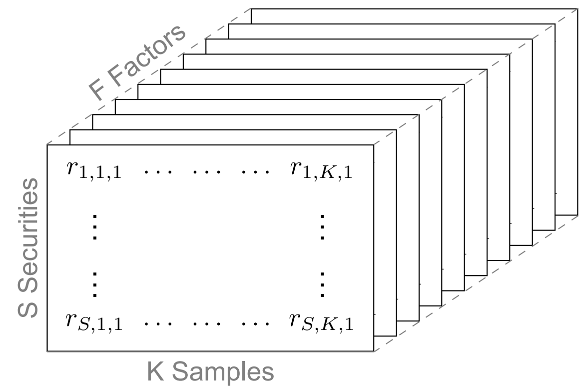

Figure 1: Results of Monte Carlo sampling with K samples for MAC on a portfolio of S securities

Each element of the 3D sample matrix r_{sfk} is the return of the s^{th} security,from the f^{th} factor, for the k^{th} Monte Carlo sample. The portfolio return distribution sample r_k^p can be computed as

$$r_k^p = \sum_i^S \sum_k^F w_s r_{sfk} \tag{20}$$

We can write this sum in two ways. Either as a weighted sum of returns of the s^{th} security from all the factors $r_{sk} = \sum_{f=1}^{M} r_{sfk}$

$$r_k^p = \sum_s^S w_i r_{sk} \tag{21}$$

Or as a sum of returns from the f^{th} factor from all the securities in portfolio $r_{fk} = \sum_{s=1}^{S} w_s r_{sfk}$

$$r_k^p = \sum_f^F r_{fk} \tag{22}$$

Portfolio VaR estimator

To compute Monte Carlo estimator of the portfolio VaR we first use formula (20) to compute all K samples r_k^p from portfolio return distribution. These samples are sorted and the estimator (11) is used to get the value of portfolio VaR for a given probability α (VaR_α) and the index of the location of the estimator VaR_α in the vector of portfolio return samples. The sorting order of the portfolio returns samples is stored - we will denote the sorted sample index $k^\star$ to distinguish it from the original index sample k. The obtained estimator for VaR_α, and its location in the sorted vector of portfolio return samples $k^\star_{VaR}$ is then used for computing marginal contributions and components VaR in both security and factor spaces.

Security contribution to VaR

To calculate security contributions to VaR we will use the formula (21) for portfolio distribution sample expressed through the returns of individual securities.

$$r^p_{k^\star} = \sum_s^S w_s r_{sk^\star}$$

where $r_{sk^\star}$ is the sum of returns of the s^{th} security from all the factors for Monte Carlo sample $k^\star$. Note the usage of the $k^\star$ index - each security distribution sample vector $\mathbf{r}_s = \{r_{sk^\star}, k^\star = 1, ..., K\}$ is sorted in the same order as the vector of portfolio returns distribution.

To compute the estimator for marginal contribution to VaR from security s using equation (16) we first need to define the averaging region boundary ε. We typically pick the width of the averaging region in formula (16) corresponding to certain percentage of the total Monte Carlo samples. For example, if we use $K = 5000$ Monte Carlo samples and define the width of the averaging region as 5% that would correspond to 250 samples in the averaging region, or the value of ε = 125. Having defined ε we compute each security s estimator for marginal contribution to VaR $MVaR_s$ as

$$\widehat{MVaR}_{\alpha,s} = -E(r_s | r_p = -VaR) = -\frac{\sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} r_{sk^\star}}{2\varepsilon} \tag{23}$$

where 2ε is the total number of sample points in the averaging region. We can then compute the normalization constant ω using equation (18) and obtain the adjusted estimator for marginal contribution to VaR from security s as

$$MVaR_{\alpha,s} = \omega \cdot \widehat{MVaR}_{\alpha,s} \tag{24}$$

Finally, the contribution to VaR from a security s is computed as

$$CVaR_s = w_s \cdot MVaR_{\alpha,s} \tag{25}$$

Factor Contribution to VaR

Computation of the marginal contribution to VaR from a given risk model factor f is similar to computation of security contribution. We start from the representation of the portfolio return distribution sample as a sum of factor returns (Eq.(22))

$$r^p_{k^\star} = \sum_f^F r_{fk\star}$$

(note again the usage of ordered portfolio sample index $k^\star$). The factor marginal contribution to VaR is then computed in the same way as security contribution, but starting from the 3D Monte Carlo return matrix aggregated along the security direction (vertical aggregation in the figure (1)). For each factor f the unadjusted estimator of the marginal contribution to VaR is computed as

$$\widehat{MVaR}_{\alpha,f} = -\frac{\sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} r_{fk^\star}}{2\varepsilon} \tag{26}$$

The normalization coefficient ω is again computed in the same way as for security contributions:

$$\omega = \frac{VaR_\alpha}{\sum\limits_f^M \widehat{MVaR}_{\alpha,f}} \tag{27}$$

And, finally, adjusted marginal factor contribution to VaR is computed as

$$\widehat{MVaR}_{\alpha,f} = \omega \cdot MVaR_{\alpha,f}$$

It is important to note that the normalization coefficients ω computed using estimators for marginal security contributions, or marginal factor contributions, are exactly the same (see Appendix()). This allows us to compute ω only once, when first set of marginal contributions is estimated, and also use the same value of ω to adjust the individual security and factor contribution estimates (see below).

Security and Factor contribution

The most granular decomposition has to be computed directly from the 3D Monte Carlo matrix. Effectively, this decomposition is based on representation of the portfolio VaR as the sum of conditional expectation of individual security returns for each individual factor

$$VaR = -\sum_{s}^{S} w_s \sum_{f}^{F} E(r_{sf}|r_p = -VaR) \tag{28}$$

Analogous to the marginal security or factor contribution, we can compute the adjusted marginal contribution from individual factor and individual security as

$$MVaR_{sf} = -\omega \frac{\sum\limits_{k=k^{\star}_{VaR}-\varepsilon}^{k=k^{\star}_{VaR}+\varepsilon} r_{sfk}}{2\varepsilon} \tag{29}$$

This results in $S \times F$ matrix of marginal contributions

$$\begin{bmatrix} MVaR_{11} & \dots & MVaR_{1F} \\ \vdots & \ddots & \vdots \\ MVaR_{S1} & \dots & MVaR_{SF} \end{bmatrix} \tag{30}$$

By construction, the sum of all element in a row of the marginal contribution matrix is the estimator of the marginal security contribution to portfolio VaR (from eq. (23)):

$$\sum_{f=1}^{F} MVaR_{sf} = -\omega \frac{\sum\limits_{k=k^{\star}_{VaR}-\varepsilon}^{k=k^{\star}_{VaR}+\varepsilon} \sum\limits_{f=1}^{F} r_{sfk}}{2\varepsilon} = -\omega \frac{\sum\limits_{k=k^{\star}_{VaR}-\varepsilon}^{k=k^{\star}_{VaR}+\varepsilon} r_{sk}}{2\varepsilon} = MVaR_s \tag{31}$$

At the same time (see eq. (26), the sum of all elements in a column of the contribution matrix is the estimator of the marginal factor contribution to portfolio VaR:

$$\sum_{f=1}^{F} MVaR_{sf} = MVaR_f \tag{32}$$

Choice of ε and numerical examples

This section analyzes numerical estimates of risk measures for a sample portfolio (a Barclays Aggregate Index). The focus is on the analysis of the errors in Monte Carlo estimates of VaR and its components.

The errors in Monte Carlo estimators arise due to statistical errors in numerical simulations of the random distributions. Monte Carlo estimators are based on the weak law of large numbers, that basically states that when the number of samples is

α	VaR_α	σ_{VaR}	$\frac{\sigma_{VaR}}{VaR_\alpha}$ %	ES_α	σ_{ES}	$\frac{\sigma_{ES}}{ES_\alpha}$ %
90	4.79	0.13	2.66	6.55	0.11	1.60
95	6.16	0.14	2.23	7.69	0.10	1.26
97	7.06	0.12	1.63	8.44	0.10	1.16
99	8.61	0.13	1.55	9.92	0.13	1.29

Table 1: Estimators and errors for VaR and ES of Barclays EUR Aggregate Index for different confidence levels α

increased towards infinity, the estimators tend towards the true values of estimated quantities. However, since in all practical applications the estimators are based on a finite number of samples, they always have an uncertainty associated with them. This uncertainty can be reduced by increasing the number of samples. In most cases the variance of the estimator that reflects that uncertainty is inversely proportional to the number of Monte Carlo samples used to obtain the estimator.

Monte Carlo process itself can also provide an estimate of the variance of the estimator. From the same computation one can obtain both estimated result and an objective measure of the statistical uncertainty in the result. In our case, we use multiple Monte Carlo simulations of a portfolio VaR and its components at different confidence levels with $K = 5000$ samples as described above. We run each simulation ten times, and record the means and standard deviations of estimated VaR and MVaR values. We use this data to evaluate the adequacy of the number of samples for our purposes and to establish an acceptable averaging region ε for computing marginal contributions to VaR that provide reasonable balance between bias and variance of the estimators.

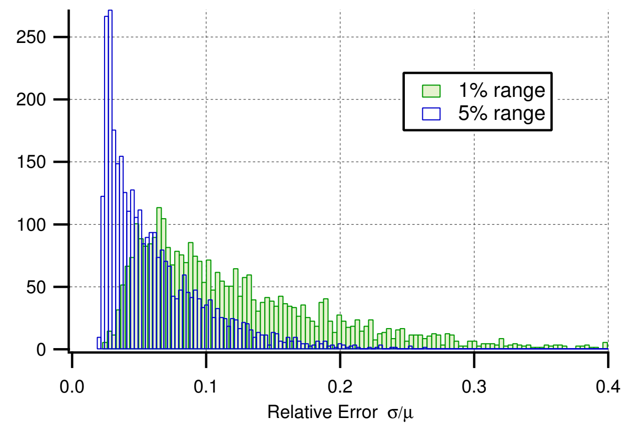

Figure 2: Distributions of component VaR Monte Carlo errors for two width of averaging regions - 1% (corresponding to 50 averaged points) and 5% (corresponding to 250 averaged points)

Table (1) shows the estimated Value-at-Risk (VaR_α) and Expected Shortfall (ES_α) values for different confidence levels α computed for Barclays EUR Aggregate Index. Also shown are Monte Carlo standard deviations of the estimators σ_{VaR} and σ_{ES} and corresponding relative error for each estimator, expressed as a percentage of the estimator itself. It is clear that with the employed number of samples ($K = 5000$) the error of both risk measures never exceeds 3%.

The index we use for testing purposes contains around 4000 securities. We analyze performance of the Monte Carlo algorithm by comparing the distributions of the errors of the components corresponding to each security obtained at different values of the averaging region width parameter ε (eq. (16)). Some of the securities have negligible impact on portfolio VaR, and the value of their corresponding components are very small. This components, if considered, will contribute large relative errors to the distribution of errors even when Monte Carlo variance is small in absolute measure. To avoid distorting the distribution of errors with this components we do not consider the components with the absolute value less then $CVaR_s < 10^{-5}$.

Figure (2) shows two such distribution for VaR_{95} obtained at values of ε corresponding to 1% and 5% of total number of samples. The distribution of errors obtained with $\varepsilon = 1\%$ (green bars on the figure) has the mean of $\mu = 0.12$ and standard deviation of $\sigma = 0.08$. In other words, most of the errors are lying below the value of 16% ($\mu + \frac{\sigma}{2}$). Increasing the value of ε to 5% of the number of samples lead to significantly narrower distribution of errors. Now the mean of the distribution is at $\mu = 6\%$, and the standard deviation is $\sigma = 4\%$, thus the majority of the errors in this case are less than 8%. Moreover, close examination of the distribution of absolute and relative errors shows that the relative errors larger than 20% are observed only on very small $CVaR$ values. In other words, only components with contributions that are not significant for the purpose of portfolio risk analysis will have large relative estimation errors.

	Mean		σ	
α	1%	5%	1%	5%
90	0.15	0.07	0.11	0.05
95	0.12	0.06	0.08	0.04
97	0.1	0.04	0.08	0.03
99	0.08	0.05	0.06	0.04

Table 2: Parameters of error distributions

Finally, Table (2) shows the mean and standard deviation of the distribution of errors obtained with different values of averaging region width parameter ε (1% and 5%) for different VaR threshold probabilities α. With the value of ε equal to 5% of the total number of samples, the center of the error distribution is located around 6%, while its width never exceed the value of 5%. Thus, the value of $\varepsilon = 5$% is sufficient to keep the errors of the VaR components within 10% range.

Conclusion

Decomposition of portfolio risk measures into components by security, asset class or factor, while relatively straightforward under the assumption of normal return distributions, becomes complicated when the normality assumption does not hold. The decomposition of risk measures requires computations of conditional expectations of component returns, conditioned on rare tail events in portfolio return distribution. The rarity of the conditioning event is expressed in relatively large errors of components estimates obtained using Monte Carlo methods. In this note we have presented the Monte Carlo methodology for portfolio risk decomposition that limits the errors of component estimates to acceptable levels. The errors can be further reduced by applying more sophisticated Monte Carlo techniques (like, for example, importance sampling) for estimating conditional expectations of tail events. These techniques, however, are not directly applicable to the case of high-dimensional risk factor model, such as MAC model, because they suffer from severe reduction of efficiency when dimensionality of the model increases to several hundreds of factors or higher (see for example [6]). Application of such techniques to risk decomposition problems in the framework of the MAC model is the subject of our current research.

References

[1] Artzner, P., et. al. (1999). Coherent Measures of Risk. Mathematical Finance, 9(3), 203-228.

[2] Hazewinkel, M. Encyclopedia of Mathematics, Vol.4. Springer (1989)

[3] Glasserman, P. (2005). Measuring Marginal Risk Contributions in Credit Portfolios. FDIC Center for Financial Research Working Papers, 1-41.

[4] Hallerbach, W. G. (2002). Decomposing portfolio value-at-risk: A general analysis. Journal of Risk, 5, 1–18.

[5] Gourieroux, C., Laurent, J. P., Scaillet, O. (2000). Sensitivity analysis of Values at Risk. Journal of Empirical Finance, 7, 225-245.

[6] Bengtsson, T., Bickel, P., Li, B. (2008). Curse-of-dimensionality revisited: Collapse of the particle filter in very large scale systems. Probability and Statistics, 2, 316-334

Marginal contribution to VaR as conditional expectation

Suppose we have a bivariate continuous random variable (X, Y) and its quantile $q(\alpha)$ defined as

$$P(X + \epsilon Y < q(\alpha)) = \alpha$$

We want to compute the derivative

$$\frac{dq(\alpha)}{d\epsilon}$$

Let's write the probability as integral over the distribution function

$$P(X + \epsilon Y < q(\alpha)) = \iint\limits_{x+\epsilon y<q} f(x,y)dxdy = \int\limits_{-\infty}^{\infty} \left[\int\limits_{-\infty}^{q-\epsilon y} f(x,y)dx \right] dy$$

Differentiating with respect to ϵ gives

$$\int\limits_{-\infty}^{\infty} \left[\frac{dq}{d\epsilon} - y \right] f(q - \epsilon y, y)dy = 0$$

leading to

$$\frac{dq}{d\epsilon} = \frac{\int\limits_{-\infty}^{\infty} yf(q-\epsilon y, y)dy}{\int\limits_{-\infty}^{\infty} f(q-\epsilon y, y)dy} = \mathbb{E}\left[Y|X + \epsilon Y = q\right]$$

Security and factor correction parameters

Let us compare the expressions for the security and factor contributions correction parameters:

$$\omega = \frac{VaR_\alpha}{\sum\limits_{s}^{M} w_s \widehat{MVaR}_{\alpha,s}} \tag{33}$$

and

$$\omega = \frac{VaR_\alpha}{\sum\limits_{f}^{M} \widehat{MVaR}_{\alpha,f}} \tag{34}$$

We can rewrite the equation for marginal security contributions (23) as

$$\widehat{MVaR}_{\alpha,s} = \frac{\sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} r_{sk^\star}}{2\varepsilon} = \frac{\sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} \sum\limits_{f=1}^{F} r_{sfk^\star}}{2\varepsilon}$$

and the equation for factor contributions Eq (26) as

$$\widehat{MVaR}_{\alpha,f} = \frac{\sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} r_{fk^\star}}{2\varepsilon} = \frac{\sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} \sum\limits_{s=1}^{S} w_s r_{sfk^\star}}{2\varepsilon}$$

Using these two expressions we can rewrite the sums of the contributions over all securities and factors that are used in the denominators of the formulas (33,34) for correction parameters:

$$\sum_{s}^{S} w_s \widehat{MVaR}_{\alpha,s} = \frac{\sum\limits_{s}^{S} \sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} \sum\limits_{f=1}^{F} w_s r_{sfk^\star}}{2\varepsilon}$$

and

$$\sum_{f}^{F} \widehat{MVaR}_{\alpha,f} = \frac{\sum\limits_{f}^{F} \sum\limits_{k=k^\star_{VaR}-\varepsilon}^{k=k^\star_{VaR}+\varepsilon} \sum\limits_{s=1}^{S} w_s r_{sfk^\star}}{2\varepsilon}$$

which shows that the sums are equal, and, therefore, the factor and security correction parameters are equal.